Harmonic generation of interfacial waves at a submerged bathymetric ridge

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[1] When two-layer, periodic, interfacial wave trains pass over a Gaussian ridge, waves of higher harmonic frequency can be excited at the ridge, which then propagate downstream independently of the incident wave. In laboratory experiments, the amount of energy transferred to higher harmonic frequencies increases with the nonlinearity of the main wave over the ridge crest. When the wave nonlinearity is increased to the point where breaking occurs over the ridge, harmonic generation can still occur. The length scale of the obstacle relative to the wavelength can also influence the energy transfer, which is consistent with analogous surface wave studies. The upper limit for which harmonic frequencies can be excited is set by the stratification. This wave-topography interaction provides a possible mechanism for transferring energy in the internal wavefield to smaller scales.


1. Introduction

[2] Internal waves have been observed in the ocean at a wide range of temporal scales. Field observations in areas with rough bathymetry do not typically show the smooth continuum of internal wave energy across frequency space suggested by Garrett and Munk [1972]. Instead, peaks of energy have been observed at the dominant tidal frequencies as well as at higher harmonics of the tidal frequencies [Kunze et al., 2002; van Haren et al., 2002]. Barotropic tides interacting with bathymetry can transfer energy to internal tides [e.g., Dushaw et al., 1995; Holloway and Merrifield, 1999; Althaus et al., 2003]. How, then, is internal wave energy converted from tides to higher frequency motions? It is well known that surface waves passing over obstacles such as breakwaters and reefs can lead to the transfer of energy to higher harmonic waves [e.g., Byrne, 1969; Rey et al., 1992; Beji and Battjes, 1993]. Despite the number of studies on this phenomenon reported in surface wave literature, as well as studies on the excitation of harmonic frequencies in a continuously stratified water column, this phenomenon has not been previously studied for periodic, interfacial waves. In this study, observations indicate that harmonic wave generation can occur when interfacial waves interact with a submerged bathymetric feature. This process illustrates one mechanism for the transfer of internal wave energy to smaller temporal and spatial scales.

2. Background

[3] In continuously stratified water, there are a number of wave-topography interactions that can generate internal waves at harmonic frequencies. Bell [1975] showed that oscillating, stratified flow over a ridge can excite harmonic wave rays that propagate from the ridge. This mechanism for transferring barotropic, tidal energy to internal waves has been investigated in the lab [Lee and Beardsley, 1974; Gostiaux and Dauxois, 2007; Peacock et al., 2008] and in the ocean [e.g., Pingree and New, 1991; Azevedo et al., 2006; Gerkema et al., 2004]. Large amplitude internal solitary waves passing over a ridge can lead to overturning and the generation of dispersive wave packets [e.g., Sveen et al., 2002] as well as the generation of second mode waves along a finite thickness interface [Hüttemann and Hutter, 2001]. This study, however, appears to be the first observation of harmonic waves generated by the interaction of periodic, interfacial waves and bathymetry (Figure 1).

[4] On the other hand, the analogous problem of harmonic wave generation for surface waves propagating over obstacles has been studied in some depth. The surface wave process is of interest in the design of breakwaters, because the conversion of energy from low-frequency waves to higher frequency waves can reduce wave impacts at the shoreline [Losada et al., 1997]. Observations of surface waves suggest that the fraction of wave energy transferred to higher harmonics increases with the nonlinearity of the shoaling wave. Studies have noted that harmonic generation of surface waves tends to increase with increasing incident...
wave amplitude \cite{Rey et al., 1992} and with the Ursell number $U_r$:

$$ U_r = \frac{a \lambda^2}{h^3} \quad (1) $$

\cite{Losada et al., 1997; Brossard et al., 2009}, where $a$ is the amplitude, $\lambda$ is the wavelength, and $h$ is the local depth. $U_r$ is the ratio of the tendency for a wave to steepen (as quantified by $a/h$) to the tendencies to disperse (as quantified by $(h/\lambda)^2$). Sinusoidal water waves occur at the limit where $U_r^{-1}$ is very small. When the tendency to steepen and disperse balance, the result is waves of permanent form such as solitary waves or periodic cnoidal waves \cite[e.g., Kundu and Cohen 2004]{Kundu and Cohen 2004}. When $U_r \geq 16$, nonlinear steepening exceeds the tendency for the wave to disperse; then a nonlinear “bore” or “surge” can form at the steepening face of the wave \cite[Korteweg and De Vries, 1895; Benjamin and Lighthill, 1954]{Benjamin and Lighthill, 1954}. In a two-layer fluid, a similar criterion can be formed, where the relevant depth is the lower layer depth over the ridge $h_r$.

The steepening of a wave over an obstacle can be described in terms of energy transfer to higher frequency components, phase-locked to the incident wave. As the wave propagates over the obstacle, though, energy can be transferred between phase-locked (or bound) components and free components that propagate independently as higher frequency waves. \cite{Ohyama and Nadaoka 1994} state that the difference in phase speeds between bound and free wave components leads to energy transfer to harmonic components as the waves pass over the obstacle. In the case of surface waves propagating over a rectangular step, \cite{Ohyama and Nadaoka 1994} found that the amplitude of the harmonic component at $2\omega_1$ varies periodically with distance along the step, with a beat length:

$$ L_2 = \frac{2\pi}{k_2 - 2k_1} \quad (2) $$

where $k_1$ is the wave number over the step of waves at the forcing frequency $\omega_1$ and $k_2$ is the wave number for free waves at $2\omega_1$. The amount of energy at $2\omega_1$ that propagates downstream of the step is set by the level of energy at the downstream edge of the step. Thus, the energy in the second harmonic is maximized when the step length is equal to $0.5L_2$ (or $1.5L_2$, $2.5L_2$, …). This formulation is based on weakly nonlinear theory, and the results of \cite{Ohyama and Nadaoka 1994} indicate that the beat distance tends to decrease when incident waves are more nonlinear. \cite{Brossard et al., 2009} also hypothesize that the transfer of energy to higher harmonics is due to the resonance of these higher components over the obstacle. Although these surface wave results were obtained over a rectangular obstacle, this mechanism may influence the transfer of energy to higher harmonic frequencies over the more gradual ridge geometry in this study. The results of \cite{Rey et al., 1992} indicate that rounding the corners of a rectangular step slightly alters the local generation of vortices, but does not significantly impact the magnitude of energy transfer to higher harmonics. In the limit of gradually varying topography, the slope of the topographic ridge $(h_0/\sigma)$, however, may have an impact on the reflection coefficient and harmonic generation at the ridge, but this effect is not explored in this study.

Linear and weakly nonlinear models of surface waves can capture the dependence of the reflection from a submerged obstacle on the incident wavelength \cite{Takano, 1960; Newman, 1965a, 1965b; Miles, 1967; Mei and Black, 1969], but not the transfer of energy to higher harmonics. Higher order Stokes-type methods have also been applied to the problem and are capable of modeling the excitement of the free harmonic at $2\omega_1$ \cite[Massel, 1983; Belibassakis and Athanassoulis, 2002]{Messel, 1983}. However, modeling the transfer of energy between higher harmonic components over the ridge and downstream requires a variable depth, fully nonlinear model that can incorporate periodic wave forcing. The numerical wave tank with nonreflective open boundaries of \cite{Ohyama and Nadaoka 1994} provides perhaps the most complete modeling of this surface wave phenomenon to date.

In this study, we investigate the extent to which the excitement of harmonics in the two-layer, interfacial wave case is similar to the surface wave case. In the two-layer case, the problem is defined by 10 dimensional parameters (Figure 1): the amplitude $a$, the wave number $k$, the standard deviation of the Gaussian ridge $\sigma$, the height of the ridge $h_0$, the layer depths $h_1$ and $h_2$, the interfacial height above the
ridge $h_r$, the interfacial thickness $\delta$, the viscosity $\nu$, and the reduced gravity $g' = g(\rho_1 - \rho_2)/\rho_0$. From these parameters, a set of independent dimensionless parameters can be formed: $kh_1$, $kh_2$, $ka$, $ah_r$, $h_0/\sigma$, $Re = \sigma(g'/\delta)^{1/2}/\nu$, and $ka$. Here the wavelength $\lambda = 2\pi/k$ and the wave frequency $\omega$ are not independent and can be related through the two-layer dispersion relation:

$$\omega^2 = g'k \frac{1}{\coth(kh_1) + \coth(kh_2)}. \tag{3}$$

There are, however, occasions when using $k$ or $\omega$ is more relevant, so both are used in this study. The dimensional parameters $\delta$, $\nu$, $\sigma$, and $h_0$ are not varied in this study. The most important dimensionless parameters are thought to be $kh_1$, $ka$, $ah_r$, and $\omega/N_{\text{max}}$, where $N = \sqrt{(-g/\rho_0)(\partial \rho/\partial z)}$. In this study, $N_{\text{max}}$ is calculated from the measured density profiles, but tends to scale with $\sqrt{4g'/\delta}$. Here $\delta$ is defined as the thickness associated with 99% of the interfacial density variation. The variable $\omega/N_{\text{max}}$ is not independent and can be formulated in terms of other variables; for example, when $kh_{1,2} \geq \pi$ in the deep water limit, $\omega/N_{\text{max}} \approx \sqrt{k\delta}/8$, whereas in the shallow water limit when $kh_{1,2} \leq \pi/10$, then $\omega/N_{\text{max}} \approx \frac{k}{2} \sqrt{\delta h_1 h_2/(h_1 + h_2)}$. For wave numbers $k_{i,j}$ and $k_{i,b}$ as well as wavelengths $\lambda_{i,j}$ and $\lambda_{i,b}$, the subscript $i$ specifies the harmonic (e.g., $i = \omega/\omega_1$), $f$ indicates a free component, and $b$ indicates a bound component.

3. Methods

Experiments were conducted in the Internal Wave Facility of the Environmental Fluid Mechanics Lab at Stanford University. Periodic, monochromatic, interfacial waves were generated by a vertically oscillating, semicylindrical wave maker at one end of a tank 480 cm long, 60 cm tall, and 30 cm wide. Waves propagate down the tank to a Gaussian ridge with an overall length of 100 cm, a standard deviation of 25 cm, and a height of 20.2 cm. At the far end of the tank, waves are absorbed by a horsehair beach.

The tank was stratified with salt (NaCl) to form two homogeneous layers separated by a thin interface. The density difference between the two layers $(\rho_1 - \rho_2)/(\rho_1 + \rho_2)/2$ was either 1% or 3%. The interface was sharpened so that 99% of the density variation occurred over the interfacial thickness $\delta = 1.5 \pm 0.15$ cm at the start of each experimental run. The lower layer depth $h_1$ was varied between 24 cm $\leq h_1 \leq 36$ cm, and the total water depth was maintained at $h_1 + h_2 = 56$ cm. The waves of a range of amplitudes were generated (0 cm $< a_{\infty} < 3$ cm) at a range of frequencies, $0.05\pi$ rad/s $< \omega < 0.8\pi$ rad/s.

Planar Laser-Induced Fluorescence (PLIF) was used to obtain images of the wave flows. The lower layer was dyed with a laser fluorescent dye, Rhodamine 6G. A scanning mirror mounted above the tank sweeps the beam of an argon-ion laser through the imaging area. Concurrently, a 12 bit, 1024 $\times$ 1024 pixel CCD camera takes a photo of the illuminated dye. Interfacial height displacements were made by fitting a tanh profile to vertical columns of the image pixel intensity. The interfacial displacement corresponds to the height $z$ where the argument of the tanh function is zero. Particle image velocimetry (PIV) was used to obtain the velocities for the Froude number measurements referenced in section 5. Additional details of the imaging system and facility can be found by Troy and Koseff [2005] and Hult et al. [2009].

4. Results

4.1. Observations of Wave Generation

When periodic trains of interfacial waves passed over the ridge, we observed that waves of higher harmonic
frequencies were generated over the ridge and then propagated downstream from the ridge, in the same direction as the main wave. The generation process of these waves is shown in Figure 2. When the trough of the wave passes over the ridge crest, a secondary crest can be seen emerging over the ridge in Figure 2c. Downstream, regular, higher frequency oscillations of the interface are visible, superimposed on the main wave. These higher frequency waves then propagate downstream as seen in Figures 2d–2i.

Figure 2 shows a case where the interface is relatively close to the ridge crest \( \frac{h_r}{x} = 5.2 \text{ cm} \), but harmonic waves were observed for deeper waves as well.

The interfacial displacement for one case of harmonic wave generation is shown in Figure 3a. Each horizontal row of the figure corresponds to the surface displacement at a given time \( t \), obtained from digital images. The alternating dark and light bands correspond to wave crests and troughs propagating in space and time. The dominant feature in Figure 3a is at the fundamental frequency, but just downstream of the ridge crest \( x = 0 \), steeper characteristics with a blurred appearance indicate the presence of higher frequency waves. In Figure 3b, a high-pass filter in time is applied to the interfacial displacement signal to remove oscillations at the fundamental frequency \( \omega_1 \) showing the paths of higher frequency waves. From Figures 3c and 3d, it is clear that energy transfer to the harmonic frequency \( 2\omega_1 \) occurs over the ridge. In other cases, additional higher harmonic frequencies were also observed depending on the incident wave frequency, as seen in Figure 4a. The lower the forcing frequency \( \frac{\omega_1}{N_{\text{max}}} \), the more higher harmonics are excited.

The energy in the fundamental frequency as well as in higher harmonics is confined to a very narrow band in frequency space, as shown in Figure 5. The observed peak width is set here by the frequency resolution \( \Delta \omega/\omega_1 = 0.05 \). Even as \( a_\omega/h_r \) is increased and waves break at the ridge, the energy in higher harmonics is still restricted to narrow frequency bands. In surface wave experiments, the observed peaks in the energy spectra were more broad, particularly in the case of breaking waves [Beji and Battjes, 1993]. In the right side of Figure 5, the interface displacement at the downstream edge of the ridge is shown for three periods of the corresponding experiments.

4.2. Wave Speed

One key question is “Are higher frequency waves bound to the main wave (i.e., phase-locked) or do they propagate independently?” Because the generated harmonic waves are dispersive, the phase speed of components can be used to distinguish whether the components are bound or free. Brossard et al. [2000] employed a moving probe technique to distinguish the Doppler-shifted spectral peaks of bound and free harmonic components on the basis of phase speed. To investigate the phase speed in this study, the interfacial height signal is band-pass filtered about the frequency of interest to remove higher and lower frequency fluctuations. Least squares regression is then used to fit a linear curve to the downstream wave characteristic, following the path of an individual wave crest in space and time. The mean slope of the curves fitted to different wave characteristics downstream of the crest is taken to be the phase speed \( c_p \) of the wave.
The wave amplitudes as well as the phase speeds of harmonic components calculated using this characteristic method are shown in Figure 4 for five cases. For the components shown in Figure 4b by (cross), the signal-to-noise ratio was greater than 4 for the peak in the energy spectrum of the corresponding amplitude shown in Figure 4a. In general, the agreement is very good between the measured and the theoretical value obtained using the two-layer, dispersion relation (equation (3)):

\[ c_p = \left( \frac{g'}{k} \frac{1}{\coth(kh_1) + \coth(kh_2)} \right)^{1/2} \text{.} \] (4)

If the higher frequency characteristics (2\(\omega_1\), 3\(\omega_1\), 4\(\omega_1\), ...) corresponded to components that were phase-locked to the fundamental wave at \(\omega_1\), then these higher frequency waves would travel at the same phase speed as the base wave, but this is not the case here. Given the agreement between the free wave dispersion relation and the measured phase speeds, it appears that the free components are larger than any bound harmonic components. This is true for all higher harmonic frequencies examined in this study. It is possible, however, that although most of the energy in each higher frequency is in the free wave component, a small percentage of the wave energy may exist in bound components. In their surface wave study, Brossard et al. [2009] found that upstream of the submerged obstacle, the bound or Stokes component at 2\(\omega_1\) was typically 1.5–4 times larger than the free component, whereas downstream of the obstacle, the transmitted free component was typically 1.5–3 times larger than the transmitted bound component, with the relative amplitudes varying with \(\omega_1\).

### 4.3. Stratification Limits on Generation

Unlike in the surface wave case, the stratification between the layers can inhibit the generation of high-frequency interfacial waves. In a stratified flow, the buoyancy frequency \(N\) is the highest frequency at which waves can propagate away from the point of generation. If a flow is perturbed at a frequency greater than \(N\), the energy will be dissipated locally. In Figure 6, the energy in harmonic frequencies \(\omega = n\omega_1\) is shown as a fraction of the total wave energy for a range of experimental conditions. It is clear that there is not significant wave energy in waves at harmonic frequencies that exceed 40% of the maximum buoyancy frequency \(N_{\text{max}}\) downstream of the ridge. No higher harmonics at frequencies above \(\omega/N_{\text{max}} = 0.5\) contain more than 0.1% of the transmitted wave energy, with the exception of one breaking wave case. For this two-layer system, the limit of \(\omega/N_{\text{max}} \leq 0.4\) can be rewritten in terms of relevant length scales as \(k\delta \leq 1.3\) for the deep wave case \((kh_{1,2} \geq \pi)\) or \(k\sqrt{h_1 h_2/(h_1 + h_2)} \lesssim 0.8\) in the shallow wave limit \((kh_{1,2} \lesssim \pi/10)\).

The two-layer density profile at rest can be approximated by the function \(\rho(z) = \rho_0 - \Delta \rho \tanh(5.3z/\delta)\) [e.g., Troy...
0.5 just above which harmonic generation occurs. This is consistent with observations of surface wave bore formation above a critical value of $U_r$ [e.g., Kundu and Cohen 2004]. Above the critical value of $U_r \approx 10$ in Figure 7a, harmonic generation tends to increase, but in some cases, very low generation is observed for high $U_r$. In Figure 7b, higher energy transfer to harmonics tends to occur for higher $a_{\infty}/h_r$; however, there is much variation in the data.

When $0.3 \leq a_{\infty}/h_r \leq 0.5$, waves at the ridge become unstable and break with more vigorous breaking as $a_{\infty}/h_r$ increases [Hult et al., 2009]. The transition to breaking may occur for a wide range $U_r$, depending on the particular conditions, as shown in Figures 7a and 8a. Although $U_r \approx 10$ seems to provide a lower bound on when harmonic generation occurs, $U_r$ does not seem to be the best predictor of the breaking transition or the magnitude of $E_{2\omega_1}/E_{\text{total}}$.

The relationship between $E_{2\omega_1}/E_{\text{total}}$ and the nonlinearity parameter becomes more clear when the data are separated by frequency, as shown in Figure 8. For high-frequency cases, when $\omega_1/N_{\text{max}}>0.21$, there is no substantial excitement of harmonic frequencies due to the stratification (see Figure 6), independent of $a_{\infty}/h_r$ or $U_r$. For intermediate frequencies, $0.06 < \omega_1/N_{\text{max}} < 0.21$, the fraction of wave energy in higher harmonics tends to increase with $a_{\infty}/h_r$ and $U_r$ prior to wave breaking. In Figures 8b(ii–iv), the relationship between $a_{\infty}/h_r$ and $E_{2\omega_1}/E_{\text{total}}$ is roughly linear for nonbreaking waves, but the rate of increase varies by frequency $\omega_1/N_{\text{max}}$. In Figure 8b(iii), the maximum energy transfer to harmonic waves occurs at $a_{\infty}/h_r \approx 0.5$ just beyond the breaking transition.

As $a_{\infty}/h_r$ increases toward the limit where the wave amplitude approaches the lower layer depth above the ridge $h_r$, the transfer of energy to harmonic waves tends to decrease in cases where $\omega_1/N_{\text{max}}>0.1$. This effect is illustrated in Figures 9b and 9c. In Figure 9a, however, there is no observed decrease in harmonic excitement. Two

**Figure 5.** (Left) Energy spectra and (right) interfacial displacement $\eta/h_r$. (a) Here $a_{\infty}/h_r = 0.1$ (not breaking), (b) $a_{\infty}/h_r = 0.25$ (near breaking), (c) $a_{\infty}/h_r = 0.6$ (mild breaking), and (d) $a_{\infty}/h_r = 0.9$ (vigorous breaking). Both the spectra and interfacial time series are obtained at the downstream edge of the ridge ($x = 50$ cm) and $\omega_1/N_{\text{max}} = 0.12$.

and Koseff 2005]. The value $N_{\text{max}}$ is the buoyancy frequency in the measured density profile, at the mid-interface level. In a continuous stratification, slightly higher values of $\omega/N$ can be supported. For example, Cacchione and Wunsch [1974] experimentally generated internal waves at frequencies up to $\omega/N = 0.74$ and Ivey et al. [2000] up to $\omega/N = 0.78$. In experiments simulating tidal flow with constant $N$ over a series of ridges, Aguilar et al. [2006] found internal waves excited at frequencies high as $\omega/N = 0.7$. Recent numerical results for tidal flow over a flat topped ridge show energy in harmonic frequencies up to and above the background $N$, but energy at frequencies above $N$ did not propagate away from the ridge as internal waves [Korobov and Lamb, 2008]. The transition value of $\omega/N$ above which harmonic excitement is limited may be sensitive to the particular stratification present.

**4.4. Wave Nonlinearity**

The degree of nonlinearity of the wave-ridge interaction clearly impacts energy transfer to higher harmonic frequencies. The roles of two parameters, the Ursell number $(U_r = a_{\infty} \lambda_{h_r})$ and the amplitude scaled by the lower layer depth $(a_{\infty}/h_r)$, are compared in Figure 7. In Figure 7a, there appears to be a cutoff value of $U_r \approx 10$ above which harmonic generation occurs. This is consistent with observations of surface wave bore formation above a critical value of $U_r$ [e.g., Kundu and Cohen 2004]. Above the critical value of $U_r \approx 10$ in Figure 7a, harmonic generation tends to increase, but in some cases, very low generation is observed for high $U_r$. In Figure 7b, higher energy transfer to harmonics tends to occur for higher $a_{\infty}/h_r$; however, there is much variation in the data.

When $0.3 \leq a_{\infty}/h_r \leq 0.5$, waves at the ridge become unstable and break with more vigorous breaking as $a_{\infty}/h_r$ increases [Hult et al., 2009]. The transition to breaking may occur for a wide range $U_r$, depending on the particular conditions, as shown in Figures 7a and 8a. Although $U_r \approx 10$ seems to provide a lower bound on when harmonic generation occurs, $U_r$ does not seem to be the best predictor of the breaking transition or the magnitude of $E_{2\omega_1}/E_{\text{total}}$.

The relationship between $E_{2\omega_1}/E_{\text{total}}$ and the nonlinearity parameter becomes more clear when the data are separated by frequency, as shown in Figure 8. For high-frequency cases, when $\omega_1/N_{\text{max}}>0.21$, there is no substantial excitement of harmonic frequencies due to the stratification (see Figure 6), independent of $a_{\infty}/h_r$ or $U_r$. For intermediate frequencies, $0.06 < \omega_1/N_{\text{max}} < 0.21$, the fraction of wave energy in higher harmonics tends to increase with $a_{\infty}/h_r$ and $U_r$ prior to wave breaking. In Figures 8b(ii–iv), the relationship between $a_{\infty}/h_r$ and $E_{2\omega_1}/E_{\text{total}}$ is roughly linear for nonbreaking waves, but the rate of increase varies by frequency $\omega_1/N_{\text{max}}$. In Figure 8b(iii), the maximum energy transfer to harmonic waves occurs at $a_{\infty}/h_r \approx 0.5$ just beyond the breaking transition.

As $a_{\infty}/h_r$ increases toward the limit where the wave amplitude approaches the lower layer depth above the ridge $h_r$, the transfer of energy to harmonic waves tends to decrease in cases where $\omega_1/N_{\text{max}}>0.1$. This effect is illustrated in Figures 9b and 9c. In Figure 9a, however, there is no observed decrease in harmonic excitement. Two
factors may contribute to the observed decline in harmonic
excitement as the amplitude $a_\infty/h_r$ is increased for breaking
wave cases. First, there are substantial qualitative differ-
cences in the wave-ridge interaction as breaking becomes
more intense. Figure 10 illustrates the qualitative features of
the wave-ridge interaction for two cases. As $a_\infty/h_r$ approaches 1, the thinned lower layer and turbulent break-
ing at the ridge may alter the process that leads to harmonic
excitement.

For the surface wave case, Beji and Battjes [1993]
suggest that the wave breaking process is secondary to the
generation of harmonic waves as waves steepen in shallow
water. Beji and Battjes observed that the distribution of
energy did not change substantially between breaking and
nonbreaking waves. However, the spectra for near-breaking
and breaking wave cases are similar in this study (Figure 5).
In the limit where $a_\infty/h_r$ is large, the breaking behavior
becomes more similar to the wave breaking events in the
simulations of Venayagamoorthy and Fringer [2007], with
pulses of mixed fluid periodically ejected over the ridge.
In this limit, it is possible that the generation of harmonics may
be altered.

Another factor contributing to the decline in harmonic
excitement as breaking intensifies may be the changing
background stratification due to wave-induced mixing. In
the stated frequency $\omega/N_{\text{max}}$, $N_{\text{max}}$ is calculated from the
background stratification at the beginning of the experiment.
In an experiment with intense breaking, $N_{\text{max}}$ can
decrease by 50%–70% over the course of a 10-wave
experiment. As a result, the effective $\omega/N$ may be sufficient
to reduce harmonic excitement, as shown in Figure 6.

For lower frequencies, it is possible that a decrease may be
observed if sufficiently large amplitude waves could be
generated. In Figure 9d, the frequency is sufficiently high
relative to $N_{\text{max}}$ to suppress any substantial excitement of
harmonics.

For the range of parameters in this study, $\omega/N_{\text{max}}$
is generally correlated with the scaled wavelength $k_\sigma$. Therefore,
the variation of $E_{2\omega}/E_{\text{total}}$ with $\omega/N_{\text{max}}$ could also be
described by the wavelength relative to the obstacle length
scale. This is addressed in section 4.5 on spatial modulation.
4.5. Spatial Modulation

Both Ohyama and Nadaoka [1994] and Brossard et al. [2009] state that the ratio of the obstacle length to a length scale of the waves impacts harmonic generation, although different wavelength scales are proposed. Ohyama and Nadaoka [1994] suggest that the excitement of the harmonic frequency \(2\omega_1\) varies periodically with the ratio of the obstacle length \(D\) to the beat length \(L_2\), as discussed in section 2. Similarly, the higher harmonic \(3\omega_1\) is expected to vary periodically with a shorter spatial scale \(D/L_3\), where \(L_3 = 2\pi/(k_3 - 3k_1)\) [Ohyama and Nadaoka, 1994]. Although their study focused on waves propagating over flat obstacles, a similar modulation seems to occur over the crest of a ridge in this two-layer study. The standard deviation of the Gaussian ridge \(\sigma\) is used for the topographic length scale of this modulation. A similar trend to the numerical results of Ohyama and Nadaoka [1994] is seen in Figure 11a in the energy of the lowest harmonic \(\omega = 2\omega_1\) for \(\sigma/L_2 < 1\). In Figure 11a, \(E_{2\omega}/E_{\text{Total}}\) peaks when \(\sigma/L_2 = 0.3\), 0.3, with low excitement when \(\sigma/L_2 = 0\) and \(\sigma/L_2 = 1\). If no other factors limited harmonic generation, a second peak in the harmonic energy would be expected, when \(1 \leq \sigma/L_2 \leq 2\); however, harmonic energy transfer is limited by the stratification when \(\sigma/L_2 > 0.8\) for \(\omega = 2\omega_1\) and when \(\sigma/L_2 > 0.2\) for \(\omega = 3\omega_1\) for this case. Brossard et al. [2009] argue that the ratio of the obstacle length to the bound harmonic wavelength \((\lambda_{3b} = 2\pi/k_{3b})\) over the ridge impacts harmonic generation in the case of waves over a submerged plate. In that study, the maximum harmonic transmission occurred for \(D/\lambda_{3b}\) between 0.7 and 1.05 and \(D/\lambda_{3f}\) between 0.9 and 1.35, depending on the water depth over the plate. In Figures 11b and 11c, the ratios \(\sigma/\lambda_{3b}\) and \(\sigma/\lambda_{3f}\) are shown and the results are consistent with the findings of Brossard et al. [2009]. Variation in the excitement of harmonic waves with the ratios \(\sigma/L_2\), \(\sigma/\lambda_{3b}\), and \(\sigma/\lambda_{3f}\) suggest that the scaled obstacle length may impact this energy transfer, but further study is needed to determine which wavelength scale controls this process.

5. Discussion

Although many of the processes observed in this study echo similar observations in surface wave experiments, some notable differences exist. The suppression of higher harmonics by the stratification in the two-layer case here is distinct, although not surprising. Also, the peak in the fraction of harmonic energy observed near the transition to breaking has not been observed in surface water experiments, although the transmission of higher harmonics under breaking surface wave conditions has received limited attention [Beji and Battjes, 1993]. Although the Ursell number has been related to harmonic generation in surface wave studies [e.g., Rey et al., 1992; Brossard et al., 2009], this study suggests that the parameter \(a_\infty/h_r\) is also useful. When breaking and nonbreaking waves are considered, Figure 8 indicates that the parameter \(a_\infty/h_r\) provides a more consistent measure of \(E_{2\omega}/E_{\text{Total}}\) than \(U_r\), beyond the transition to wave breaking.

The fate of incident wave energy was investigated by comparing the interface displacement on both sides of the ridge. The energy transmitted at various frequencies, the energy associated with the breaking event, and the energy...
transferred to viscous damping as a percentage of the upstream wave energy can be estimated by comparing the interfacial displacement upstream of the ridge at \( x_u = -64 \text{ cm} \) with the signal downstream at \( x_d = 85 \text{ cm} \). Applying the theory developed by Troy and Koseff [2006], the percentage of the upstream wave energy transferred to viscous damping not associated with breaking, such as damping at the sidewalls, bottom, interior, and interface, is approximately 24% between \( x_u \) and \( x_d \) for this frequency. Once the transmitted wave energy and nonbreaking viscous damping are accounted for, the remaining difference in wave energy between the \( x_u \) and \( x_d \) locations is assumed to be associated with the wave breaking event, which includes both turbulent kinetic energy dissipation and irreversible mixing. For a weak breaking event where \( a_{\infty}/h_r = 0.5 \) and \( \omega_1/N_{\text{max}} = 0.14 \), 9% of the upstream wave energy is transmitted at the harmonic frequency \( 2\omega_1 \), whereas 14% ± 5% of the upstream wave energy is associated with the wave breaking event leading to turbulent kinetic energy dissipation and irreversible mixing. As \( a_{\infty}/h_r \) is increased, harmonic excitement tends to decrease while the energy transferred to dissipation and irreversible mixing in the breaking event tends to increase. For the more vigorous breaking event where \( a_{\infty}/h_r = 0.85 \) shown in Figure 10b, 0.3% of the upstream wave energy is transmitted at \( 2\omega_1 \) and 44% ± 5% is associated with the breaking event.

[28] The experimental Reynolds number \( (Re \approx 10^2 - 10^4) \) is lower than it would be at oceanic scales \( (Re \approx 10^5 - 10^7) \), so viscous damping of interfacial waves will be more prominent in these experimental results. The rate of viscous decay increases with frequency [Troy and Koseff, 2006], so higher frequency waves may be damped significantly before reaching the downstream measurement location. The percentage of energy lost to viscous damping is shown in Figure 12 for nonbreaking waves between the ridge crest \( (x = 0 \text{ cm}) \) and the measurement location \( (x = 50 \text{ cm}) \). Over the ridge crest, higher frequency components may make up a slightly larger fraction of total wave energy than at the downstream location.

[29] A number of mechanisms influencing the magnitude of harmonic generation have been discussed here: stratification strength \( (\omega/N_{\text{max}}) \), incident wave nonlinearity \( (a/h_r, U_r, U_i) \), obstacle versus wavelength scales \( (\sigma/L_2) \), and viscous wave dissipation. Some of the variation in the strength of harmonic excitation could be explained by more than one of these proposed factors. For example, Figure 13 shows that the decline in \( E_{2\omega_1}/E_{\text{Total}} \) as \( \sigma/L_2 \) approaches 1 coincides with the proposed limit of \( (\omega/N_{\text{max}} \approx 0.4) \). Because of the correlation between \( 2\omega_1/N_{\text{max}} \) and \( \sigma/L_2 \) for the conditions explored in this study, it is not obvious whether the stratification or the length scale ratio is limiting harmonic generation. To examine the dependence of transmitted harmonics on \( \sigma/L_2 \) independently of the stratification limit, one could vary the obstacle length for a fixed wavelength and frequency. Because all the generation of all higher harmonics is limited when \( \omega/N_{\text{max}} \approx 0.4 \), there is more evidence in this study to support the stratification limit than the modulation based on \( \sigma/L_2 \), but both factors may be important.

[30] In this study, hydraulic control is not thought to be important in the generation of harmonic waves. If steady, two-layer hydraulic theory is used to calculate a composite Froude number, \( G^2 = Fr_1^2 + Fr_2^2 \), where \( Fr_i = u_i/\sqrt{g/h_i} \) for \( i = 1, 2 \), then PIV measurements for \( u_1 \) and \( u_2 \) were used to show that \( G < 1 \) at all times in all nonbreaking wave cases [see Hult et al., 2009]. Because \( G < 1 \) everywhere, there is not a control point over the ridge, and thus internal hydraulic control is not integral to the generation of these harmonic waves. Although the steady theory is not appropriate for hydraulic control due to the interaction of waves

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**Figure 11.** Energy transfer to higher harmonics \( 2\omega_1 \) (open circle) and \( 3\omega_1 \) (triangle) for waves of varied forcing frequency \( \omega_1 \). \( L_2 \) is defined in equation (2) and \( \lambda_{2,b}, \lambda_{2,f} \) are calculated over the ridge crest using equation (3).

**Figure 12.** Expected decay based on Troy and Koseff [2006] between the ridge crest \( (x = 0 \text{ cm}) \) and the measurement location \( (x = 50 \text{ cm}) \). The symbol size indicates the magnitude of \( E_{2\omega_1}/E_{\text{Total}} \) and the legend shows 10% for scale.
of similar time scales, it is not thought that hydraulic control is a plausible explanation for wave generation here. There are cases when low-frequency internal or baroclinic motions over topography can excite higher frequency waves [e.g., Apel et al., 1985], but in this study the phasing of wave generation (see Figure 2) and estimated Froude numbers are not consistent with this mechanism.

6. Conclusions

[31] When interface waves propagate over a ridge, energy can be transferred to higher frequency waves which then propagate downstream, independently of the main wave as free components. There is minimal energy transfer to higher frequency components that are phase-locked to the main wave. The measured phase speed of higher frequency components is consistent with the two-layer dispersion relation. The generation of harmonic waves over the ridge is limited by the stratification when the harmonic frequency exceeds approximately 40%–50% of the maximum buoyancy frequency. Below this stratification limit, higher harmonics at up to 10 times the incident wave frequency were observed, and the energy in higher harmonic components tends to decrease with the frequency. Energy transfer to harmonic waves may also be modulated by the length of the topographic ridge relative to the length scale of the wave.

[32] Harmonic generation is observed in cases when \( \omega / N_{\text{max}} < 0.5 \) and when \( U_r > 10 \). The fraction of wave energy in higher frequency components increases with the linearity of the incident wave, as quantified by \( a_{\infty} / h \) and \( U_r \). Although harmonic waves can be excited in breaking and nonbreaking cases, the energy transfer is maximized at or just beyond the critical amplitude where wave breaking occurs (0.4 \( \leq a_{\infty} / h \leq 0.6 \)). Beyond this amplitude, harmonic wave generation tends to decrease. Interactions between periodic wave trains and a ridge can lead to as much as 40% of the transmitted wave energy propagating as higher harmonic waves.

References


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